EXAMPLES OF MOTIVIC COHOMOLOGY OF CLASSIFYING SPACES II (Cohomology Theory of Finite Groups and Related Topics)

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EXAMPLES OF MOTIVIC COHOMOLOGY OF CLASSIFYING SPACES (II)

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1. INTRODUCTION

Let G be a compact Lie group. Taking complexification, we can identify the group $G=G_{\mathbb{C}}$ as the reductive algebraic group over the complex number field $\mathbb{C}$. The main result of this paper is the computation of the mod p motivic cohomology $H^{*,*'}(BG;\mathbb{Z}/p)$ of the classifying spaces of algebraic groups (over $\mathbb{C}$) corresponding Lie groups G. We compute $H^{*,*'}(BG;\mathbb{Z}/2)$ for $G=O_n$, SO_4, Q_8 and D_8.

2. THE MOTIVIC COHOMOLOGY OF $B\mathbb{Z}/p$

In this section we consider the relation between the motivic and the usual ordinary cohomologies. Let R be $\mathbb{Z}$ or $\mathbb{Z}/p$. The motivic cohomology has the following properties ([6],[7],[9]).

(C1) $H^{*,*'}(X;R)$ is a bigraded multiplicative cohomology theory in (some good) category *Spc* of pointed (algebraic) spaces (the cohomology of a space means the reduced cohomology of a pointed space); For any map $f:X\to Y$ in the category *Spc*, we have the cofiber sequence $X\to Y\to Y/X$, which induces the long exact sequence

$$\leftarrow H^{*,*'}(X;R)\leftarrow H^{*,*'}(Y;R)\leftarrow H^{*,*'}(Y/X:R)\leftarrow H^{*-1,*'}(X;R)\leftarrow ...$$

(In particular, we get the Mayer-Vietoris, Gysin and blow up long exact sequences.)

(C2) There are maps (realization maps)

$$t_{\mathbb{C}}^{m,n}:H^{m,n}(X;R)\to H^m(X(\mathbb{C});R)$$

which sum up $t_{\mathbb{C}}^{*,*'}=\oplus_{m,n}t_{\mathbb{C}}^{m,n}$ the natural ring homomorphism.

(C3) There are (the Bockstein, the reduced powers) operations

$$\beta:H^{*,*'}(X;\mathbb{Z}/p)\to H^{*+1,*'}(X;\mathbb{Z}/p)$$

$$P^i:H^{*,*'}(X;\mathbb{Z}/p)\to H^{*+2(p-1)i,*'+(p-1)i}(X;\mathbb{Z}/p)$$

which commutes with the realization map $t_{\mathbb{C}}$.

(C4) For the projective space $\mathbb{P}^n$, there is an isomorphism

$$H^{*,*}(X\wedge(\mathbb{P}^n/\mathbb{P}^{n-1});R)\cong H^{*,*}(X;R)\{y'\}$$

with $deg(y')=(2n,n)$ and $t_{\mathbb{C}}(y')\neq 0$.

(C5) For a smooth X, if $H^{m,n}(X;R)\ncong 0$, then

$$m\leq n+dim(X),\ m\leq 2n\ \ and\ \ m\geq 0.$$

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For an element $x \in H^{m,n}(X;\mathbb{Z}/p)$, we define the (weight and difference degree)

$$w(x) = 2n - m, \quad d(x) = m - n$$

Hence for smooth X, $w(x) \geq 0$, and $d(x) \leq dim(X)$ for nonzero $x \in H^{*,*'}(X;\mathbb{Z}/p)$.

Remark. For $x' \in H^*(X(\mathbb{C});\mathbb{Z}/p)$, we can define the weight degree $w(x')$ which is the least number of $w(x)$ such that $t_{\mathbb{C}}(x) = x'$.

Lichtenbaum defined the similar cohomology $H_L^{*,*'}(X;R)$ by using the étale topology, while $H^{*,*'}(X;R)$ is defined by using Nisnevich topology. Since Nisnevich covers are some restricted étale covers, there is the natural (cyclic) map $cl^{*,*'}$: $H^{*,*'}(X;R) \to H_L^{*,*'}(X;R)$. We say that the condition $BL(n,p)$ holds if

$$BL(n,p) \quad : \quad H^{m,n}(X;\mathbb{Z}_{(p)}) \cong H_L^{m,n}(X;\mathbb{Z}_{(p)}) \quad \text{for all } m \leq n+1$$
$$(\text{hence } H^{m,n}(X;\mathbb{Z}/p) \cong H_L^{m,n}(X;\mathbb{Z}/p) \quad \text{for all } m \leq n)$$

and all smooth X. The Beilinson-Lichtenbaum conjecture is that $BL(n,p)$ holds for all n, p. It is proved that the $BL(n,p)$ condition is equivalent the Bloch-Kato conjecture (BK) for degree n and prime p. Recently, V.Voevodsky proved the Bloch-Kato conjecture [10]. Hence $BL(n,p)$ holds for all n and p.

Moreover Suslin-Voevodsky proves $H_L^{m,n}(X;\mathbb{Z}/p) \cong H_{et}^m(X;\mu_p^{\otimes n})$.

From the dimensional condition (C5) and the above isomorphism, we have isomorphisms

$$H^{m,n}(Spec(k);\mathbb{Z}/p) =$$
$$H^{m,n}(pt;\mathbb{Z}/p) \cong H_{et}^m(pt;\mu_p^{\otimes n}) \cong H_{et}^m(pt;\mathbb{Z}/p) \qquad if\ m \leq n$$

and $H^{m,n}(pt;\mathbb{Z}/p) \cong 0$ otherwise. Let $\tau \in H^{0,1}(pt;\mathbb{Z}/p)$ be the element corresponding a generator of $H_{et}^0(pt;\mu_p) \cong \mathbb{Z}/p$. Then we get the isomorphism

$$H^{*,*'}(Spec(k);\mathbb{Z}/p) \cong H_{et}^*(Spec(k);\mathbb{Z}/p) \otimes \mathbb{Z}/p[\tau]$$

since $\tau : H_{et}^m(pt;\mu_p^{\otimes n}) \cong H_{et}^m(pt;\mu_p^{\otimes(n+1)})$. For examples, with $deg(\rho) = (1,1)$,

$$H^{*,*'}(Spec(\mathbb{R});\mathbb{Z}/2) \cong \mathbb{Z}/2[\rho,\tau], \quad H^{*,*'}(Spec(\mathbb{C});\mathbb{Z}/p) \cong \mathbb{Z}/p[\tau].$$

Next we compute the motivic cohomology of $\mathbb{P}^\infty$ and $B\mathbb{Z}/p$. By the cofiber map $\mathbb{P}^{n-1} \to \mathbb{P}^n \to \mathbb{P}^n/\mathbb{P}^{n-1}$ and (C4), we can inductively prove that

$$H^{*,*'}(\mathbb{P}^\infty;\mathbb{Z}/p) \cong H^{*,*'}(pt.;\mathbb{Z}/p) \otimes \mathbb{Z}/p[y]$$

with $deg(y) = (2,1)$. The Lens space is identified with the sphere bundle associated with the line bundle. This induces the ring isomorphism for $p = odd$

$$H^{*,*'}(B\mathbb{Z}/p;\mathbb{Z}/p) \cong \mathbb{Z}/p[y] \otimes \Lambda(x) \otimes H^{*,*'}(pt;\mathbb{Z}/p)$$

with $deg(x) = (1,1)$. However note that when $p = 2$, we see ([8])

$$x^2 = y\tau + x\rho$$

where $\rho \in H^{1,1}(pt;\mathbb{Z}/p) \cong k^*/k^{2*}$ represents -1.

By the above cofiber sequence, we can easily see that $\mathbb{P}^\infty$ and $B\mathbb{Z}/p$ satisfy the Kunneth formula for all spaces (while Kunneth formula does not hold for general X, Y in the mod p motivic cohomology). In particular, we have the ring isomorphisms

$$H^{*,*'}((\mathbb{P}^\infty)^n;\mathbb{Z}/p) \cong \mathbb{Z}/p[y_1,...,y_n] \otimes H^{*,*'}(pt;\mathbb{Z}/p)$$
$$H^{*,*'}((B\mathbb{Z}/p)^n;\mathbb{Z}/p) \cong \mathbb{Z}/p[y_1,...,y_n] \otimes \Lambda(x_1,...,x_n) \otimes H^{*,*'}(pt;\mathbb{Z}/p)$$

(when $p = 2$, $x_i^2 = y_i\tau + x_i\rho$).

This fact is used to define the reduced power operation P^i in (C3). Since a Sylow p subgroup of the symmetric group S_p of p-letters is isomorphic to $\mathbb{Z}/p$, we know the isomorphism

$$H^{*,*'}(BS_p;\mathbb{Z}/p) \cong H^{*,*'}(B\mathbb{Z}/p;\mathbb{Z}/p)^{F_p^*} \cong \mathbb{Z}/p[Y] \otimes \Lambda(W) \otimes H^{*,*'}(pt;\mathbb{Z}/p)$$

with identifying $Y = y^{p-1}$ and $W = xy^{p-2}$. If X is smooth (and suppose p is odd to simplify arguments), we can define the reduced powers (of Chow rings) as follows. Consider maps

$$H^{2*,*}(X;\mathbb{Z}/p) \xrightarrow{i_!} H^{2p*,p*}(X^p \times_{S_p} ES_p) \xrightarrow{\Delta^*}$$

$$H^{2*,*}(X \times BS_p;\mathbb{Z}/p) \cong H^{2*,*}(X;\mathbb{Z}/p) \otimes_{H^{2*,*}(pt;\mathbb{Z}/p)} H^{2*,*}(BS_p;\mathbb{Z}/p)$$

where $i_!$ is the Gysin map for p-th external power, and Δ is the diagonal map. For $deg(x) = (2n,n)$, the reduced powers are defined as

$$\Delta^* i_!(x) = \sum P^i(x) \otimes Y^{n-i} + \beta P^i(x) \otimes WY^{n-i-1}.$$

Hence note $deg(P^i) = deg(Y^i) = deg(y^{i(p-1)}) = (2i(p-1), i(p-1))$.

Voevodsky defined $i_!$ for non smooth X also. By using suspensions maps, he defined reduced powers for all degree elements in $H^{*,*}(X;\mathbb{Z}/p)$ for all X [8]. Thus we get the operations in (C3).

Moreover Voevodsky defined the motivic Milnor operation such that $Q_i = [Q_{i-1}, P^{p^{i-1}}]\ mod(\rho)$ (for details see [8])

$$Q_i : H^{*,*'}(X;\mathbb{Z}/p) \to H^{*+2p^i-1,*'+p^i-1}(X;\mathbb{Z}/p)$$

which is derivative, $Q_i(xy) = Q_i(x)y + xQ_i(y)$ if $\rho = 0$. For the case $\rho \neq 0$ see [14] or [7].

3. MOTIVIC COHOMOLOGY OF BO_n AND BSO_4.

The motivic cohomology of the classifying space is defined as follows. Let G be a linear algebraic group over k. Let V be a representation of G such that G acts freely on $V - S$ for some closed subset S. Then $(V-S)/G$ exists as a quasi-projective variety over k. According to Totaro ([?]) and V.Voevodsky ([6]), we define

$$H^{*,*'}(BG;\mathbb{Z}/p) = \lim_{dim(V),codim(S)\to\infty} H^{*,*'}((V-S)/G;\mathbb{Z}/p).$$

We still know the motivic cohomologies of $B\mathbb{G}_m$ and $B\mathbb{Z}/p$. Since BGL_n is cellular, we have

$$H^{*,*'}(BGL_n;\mathbb{Z}/p) \cong \mathbb{Z}/p[c_1,...,c_n] \otimes H^{*,*'}(pt;\mathbb{Z}/p)$$

where the Chern class c_i with $deg(c_i) = (2i,i)$ are identified with the elementary symmetric polynomial in $H^{2*,*}((\mathbb{P}^\infty)^n;\mathbb{Z}/p)$. So we can define the Chern class $\rho^*(c_i) \in H^{2*,*}(BG;\mathbb{Z}/p)$ for each representation $\rho : G \to GL_n$.

Hereafter we *assume* $k = \mathbb{C}$ throughout this paper.

The mod 2 cohomology of the classifying space BO_n of the n-th orthogonal group is

$$H^*(BO_n;\mathbb{Z}/2) \cong H^*((B\mathbb{Z}/2)^n;\mathbb{Z}/2)^{S_n} \cong \mathbb{Z}/2[w_1,...,w_n]$$

where S_n is the n-th symmetry group, w_i is the Stiefel-Whiteney class which restricts the elementary symmetric polynomial in $\mathbb{Z}/2[x_1,...,x_n]$. Each element w_i^2 is represented by Chern class c_i of the induced representation $O(n) \subset U(n)$. Let us write w_i^2 by c_i.

Since $Q_{i-1}...Q_0(w_i) \neq 0$, we see each $w(w_i) = i$. However even the module structure of $gr^*H^*(BO_n;\mathbb{Z}/2)$ seems complicated. W.S.Wilson ([11],[1]) found a good $Q(i) = \Lambda(Q_0, ..., Q_i)$-module decomposition for BO_n, namely,

$$H^*(BO_n;\mathbb{Z}/2) = \oplus_{i=-1}Q(i)G_i \quad with\ Q_0...Q_iG_i \in \mathbb{Z}/2[c_1, ..., c_n].$$

Here G_{k-1} is quite complicated, namely, it is generated by symmetric functions

$$\Sigma x_1^{2i_1+1}...x_k^{2i_k+1}x_{k+1}^{2j_1}...x_{k+q}^{2j_q}, \quad k+q \leq n,$$

with $0 \leq i_1 \leq ... \leq i_k$ and $0 \leq j_1 \leq ... \leq j_q$; and if the number of j equal to j_u is odd, then there is some $s \leq k$ such that $2i_s + 2^s < 2j_u < 2i_s + 2^{s+1}$. We can prove $w(G_i) = i + 1$ and hence ;

Theorem 3.1. *An element $x \in H^*(BO_n;\mathbb{Z}/2)$ is $w(x) = s$ if and only if s is the maximal number such that $Q_{i_1}...Q_{i_s}(x) \neq 0$ for some $(i_1, .., i_s)$. Moreover we have the isomorphisms*

$$H^{*,*'}(BO_n;\mathbb{Z}/2) \cong H^{*,*'}((B\mathbb{Z}/2);\mathbb{Z}/2)^{S_n} \cong \mathbb{Z}/2[\tau] \otimes (\oplus Q(i)G_i).$$

When $n = odd$, it is well known that there is the isomorphism $O_n \cong SO_n \times \mathbb{Z}/2$. Hence we have the isomorphism

$$H^{*,*'}(BSO_{2m+1};\mathbb{Z}/2) \cong \mathbb{Z}/2[\tau] \otimes (\oplus Q(j)G'_j) \quad with\ \ G'_j = i^*G_j$$

where $i : SO_n \to O_n$ is the inclusion. (Note $p^*i^*(w_n) \neq w_n \in H^*(BO_n;\mathbb{Z}/2)$ for the projection $p : O_n \to SO_n$.)

Since the direct decomposition of BO_3 is complicated to write, we only write here that of SO_3 (note $O_3 \cong SO_3 \times \mathbb{Z}/2$).

$$H^*(BSO_3;\mathbb{Z}/2) \cong \mathbb{Z}/2[w_1, w_2, w_3]/(w_1) \cong \mathbb{Z}/2[w_2, w_3]$$

$$\cong \mathbb{Z}/2[c_2, c_3]\{1, w_2, w_3 = Q_0w_2, w_2w_3 = Q_1w_2\}$$

$$\cong \mathbb{Z}/2[c_2, c_3]\{w_2, Q_0w_2, Q_1w_2, c_3 = Q_0Q_1w_2\} \oplus \mathbb{Z}/2[c_2]$$

$$\cong \mathbb{Z}/2[c_2, c_3] \otimes Q(1)\{w_2\} \oplus \mathbb{Z}/2[c_2].$$

Of course, this case $w(w_2) = 2$ and we have

$$H^{*,*'}(BSO_3;\mathbb{Z}/2) \cong \mathbb{Z}/2[\tau] \otimes (\mathbb{Z}/2[c_2, c_3] \otimes Q(1)\{w_2\} \oplus \mathbb{Z}/2[c_2]).$$

For $n = even$, $O_n \ncong SO_n \times \mathbb{Z}/2$. The motivic cohomology seems difficult to compute. Even $n = 4$ it seems complicated. In fact, the realization map $t_{\mathbb{C}}$ is not injective (i.e., $\tau \times y_2 = t_{\mathbb{C}}(y_2) = 0$ in the following theorem).

Theorem 3.2. *The motivic cohomology $H^{*,*'}(BSO_4;\mathbb{Z}/2)$ is isomorphic to*

$$\mathbb{Z}/2[c_2, c_4]\{y_2\} \oplus \mathbb{Z}/2[\tau, c_2] \otimes (\mathbb{Z}/2[c_4]\{1\}$$

$$\oplus\mathbb{Z}/2[c_3] \otimes Q(1)\{w_2\} \oplus \mathbb{Z}/2[c_4] \otimes (\mathbb{Z}/2[c_3]Q(2) - \mathbb{Z}/2\{1\})\{a\})$$

where a is a virtual element so that $t_{\mathbb{C}}(c_3a) = w_2w_3w_4$, $t_{\mathbb{C}}(Q_0a) = w_4$, $t_{\mathbb{C}}(Q_1a) = w_2w_4$ and $t_{\mathbb{C}}(Q_2a) = c_4w_2$.

4. MOTIVIC COHOMOLOGY OF BD_8 AND BQ_8

In this section, we compute the *mod*(2) motivic cohomology of BD_8 and BQ_8.

At first, we consider the case Q_8. The mod 2 (usual) cohomology is well known (see Theorem 2.7)

$$H^*(BQ_8;\mathbb{Z}/2) \cong \mathbb{Z}/2\{1, x_1, y_1, x_2, y_2, w\} \otimes \mathbb{Z}/2[c_2]$$

where $x_i^2 = \beta x_i = y_i$ and $|w| = 3$. The graded algebra $gr^{*'}H^*(BQ_8;\mathbb{Z}/2)$ is given by letting the weight degree by

$$w(y_i) = w(c_2) = 0, \qquad w(x_i) = w(w) = 1.$$

The facts $w(y_i) = w(c_2) = 0$ follows from that they are Chern classes. We can prove that $w(w) = 1$ (in fact, we can take $w \in H^{3,2}(BQ_8;\mathbb{Z}/2)$.)

Theorem 4.1. *We have the bidegree isomorphism*

$$H^{*,*'}(BQ_8;\mathbb{Z}/2) \cong \mathbb{Z}/2[\tau] \otimes gr^{*'}H^*(BQ_8;\mathbb{Z}/2).$$

Now we consider the case $G = D_8$. We recall the *mod*(2) cohomology.

$$H^*(BD_8;\mathbb{Z}/2) \cong (\mathbb{Z}/2[x_1, x_2]/(x_1x_2)) \otimes \mathbb{Z}/2[u] \cong$$

$$(\oplus_{i=1}^2 \mathbb{Z}/2[y_i]\{y_i, x_i, y_iu, x_iu\} \oplus \mathbb{Z}/2\{1, u\}) \otimes \mathbb{Z}/2[c_2]$$

Here we identify, $y_i = x_i^2$ and $c_2 = u^2$. The cohomology operations on $H^*(BD_8;\mathbb{Z}/2)$ is well known, e.g., (see [Te-Ya])

$$Q_0(u) = (x_1 + x_2)u = e, \quad Q_1Q_0(u) = (y_1 + y_2)c_2.$$

Lemma 4.2. *There exist $u_1', u_2' \in H^{3,2}(BD_8;\mathbb{Z}/2)$ with $\tau u_i' = x_iu \in H^{3,3}(BD_8;\mathbb{Z}/2)$ (so $u_i' = \tau^{-1}x_iu$).*

Therefore we get $gr^{*'}H^*(BD_8;\mathbb{Z}/2)$ which is isomorphic to

$$(\oplus_{i=1}^2 \mathbb{Z}/2[y_i]\{y_i, x_i, x_iu_i', u_i'\} \oplus \mathbb{Z}/2\{1, u\}) \otimes \mathbb{Z}/2[c_2]$$

with $w(y_i) = w(c_2) = 0$, $w(x_i) = w(u_i') = 1$ and $w(u) = w(x_iu_i') = 2$. (Note $u, x_iu_i' \notin CH^*(BG)/2$, and $x_iu_i' = y_iu$).

Theorem 4.3. *We have the the bidegree module isomorphism*

$$H^{*,*'}(BD_8;\mathbb{Z}/2) \cong \mathbb{Z}/2[\tau] \otimes gr^{*'}H^*(BD_8;\mathbb{Z}/2).$$

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